chapter 3

Probability 2

Section 3.3 Binomial distribution – Bernoulli trials

PROJECT MATHS
Text & Tests 5
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A coin is biased in such a way that the probability of a head is always $\frac{2}{5}$.

Robbie tosses the coin four times. He wants to know the probability that there will be three heads and one tail.

The 3 heads and 1 tail can be arranged in **four** different ways.

$$P(\mathrm{H, H, H, T}) = \frac{2}{5} \times \frac{2}{5} \times \frac{2}{5} \times \frac{3}{5} = \left(\frac{2}{5}\right)^3 \times \frac{3}{5}$$

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$$P(\mathrm{T, H, H, H}) = \frac{3}{5} \times \frac{2}{5} \times \frac{2}{5} \times \frac{2}{5} = \left(\frac{2}{5}\right)^3 \times \frac{3}{5}$$

$P\left(\text{3 heads in 4 throws}\right) = ?$

The total probability for 3 heads and 1 tail $= 4 \times \left(\frac{2}{5}\right)^3 \times \frac{3}{5} = \frac{96}{625}$

Notice that the 4 in the answer is the value of $\binom{4}{3}$ and is the number of selections of 3 heads from 4 coins.

Thus the probability of 3 heads and 1 tail $= \binom{4}{3}\left(\frac{2}{5}\right)^3 \times \frac{3}{5}$

The example above is a special type of probability model called the **binomial distribution**.

A binomial distribution can be used in any experiment that has these 4 characteristics:

- A fixed number, n, of trials are carried out
- Each trial has two possible outcomes: success or failure
- The trials are independent
- The probability of success in each trial is constant.
  The probability of a success is generally called p.
  The probability of a failure is q, where $p + q = 1$.

In general, the probability of r successes in n trials is given by the formula on the right, where p is the probability of success and q is the probability of failure.

$$P(r \text{ successes}) = \binom{n}{r} p^r q^{n-r}$$

Experiments which satisfy the four conditions listed above are also called **Bernoulli Trials** after the Swiss mathematician James Bernoulli (1654–1705).

Consider the event of obtaining a 6 from a single throw of an unbiased die.

$$P(\text{success}) = \tfrac{1}{6} \quad \text{and} \quad P(\text{failure}) = \tfrac{5}{6}$$

If there are 8 such trials, then the probability of 0, 1, 2, 3, … successes from 8 attempts is given by the terms of the expansion of

$$\left(\tfrac{5}{6} + \tfrac{1}{6}\right)^8$$

Since the probability of r successes is given by $\binom{n}{r} p^r q^{n-r}$

(i) $P(\text{no six}) = \binom{8}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^8 = \binom{8}{0}\left(\frac{5}{6}\right)^8$

(ii) $P(1 \text{ six}) = \binom{8}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^7$

(iii) $P(2 \text{ sixes}) = \binom{8}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^6$

....

$P(8 \text{ sixes}) = \binom{8}{8}\left(\frac{1}{6}\right)^8\left(\frac{5}{6}\right)^0 = \binom{8}{8}\left(\frac{1}{6}\right)^8$

Example 1

An unbiased die is thrown 5 times. Find the probability of obtaining
(i) 1 six (ii) 3 sixes (iii) at least 1 six.

(i) exact way of getting 1 6 in 5 throws

$P(6, 9, 9, 9, 9) = (\frac{1}{6})(\frac{5}{6})(\frac{5}{6})(\frac{5}{6})(\frac{5}{6})$

5 other ways.

$\Rightarrow P(1 \text{ six in } 5 \text{ throws}) = 5 \times (\frac{1}{6})(\frac{5}{6})(\frac{5}{6})(\frac{5}{6})(\frac{5}{6})$

$= \frac{3125}{7776}$

$P(r \text{ successes}) = \binom{n}{r} p^r q^{n-r}$

$p = P(6) = \frac{1}{6}$
$q = P(\text{not } 6) = \frac{5}{6}$
$r = \text{no. } 6\text{s} = 1$
$n = \text{no. trials} = 5$

$P(1\ 6 \text{ in } 5 \text{ throws}) = \binom{5}{1}(\frac{1}{6})^1(\frac{5}{6})^4 = \frac{3125}{7776}$

Example 1

An unbiased die is thrown 5 times. Find the probability of obtaining
(i) 1 six (ii) 3 sixes (iii) at least 1 six.

$P(r \text{ successes}) = \binom{n}{r} p^r q^{n-r}$

$P(3 \text{ sixes}) - ?$

$= \binom{5}{3}(\frac{1}{6})^3(\frac{5}{6})^2$

$= \frac{125}{388}$

$p = P(6) = \frac{1}{6}$
$q = P(\text{not } 6) = \frac{5}{6}$
$r = \text{no. } 6\text{s} = 3$
$n = \text{no. trials} = 5$