Complex numbers

chapter 3

Section 3.10 Applications of de Moivre's Theorem

PROJECT MATHS

Text & Tests 6

Simplifying expressions of the form $(\cos\theta - i\sin\theta)^n$

Example 1

Simplify $\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)^6$, giving your answer in rectangular form. $a+bi$

Apply de Moivre

$r = 1$

$z^n = r^n(\cos n\theta + i\sin n\theta)$

$z^n = r^n(\text{cis } n\theta)$

$-\sin(A) = \sin(-A)$

$\cos(A) = \cos(-A)$

$z = 1\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$

$= 1\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$

$z^6 = 1^6\left(\cos\left(-\frac{\cancel{6}^{2}\pi}{\cancel{3}}\right) + i\sin\left(-\frac{\cancel{6}^{2}\pi}{\cancel{3}}\right)\right)$

$= 1(1 + i0)$

$= 1 + 0i$

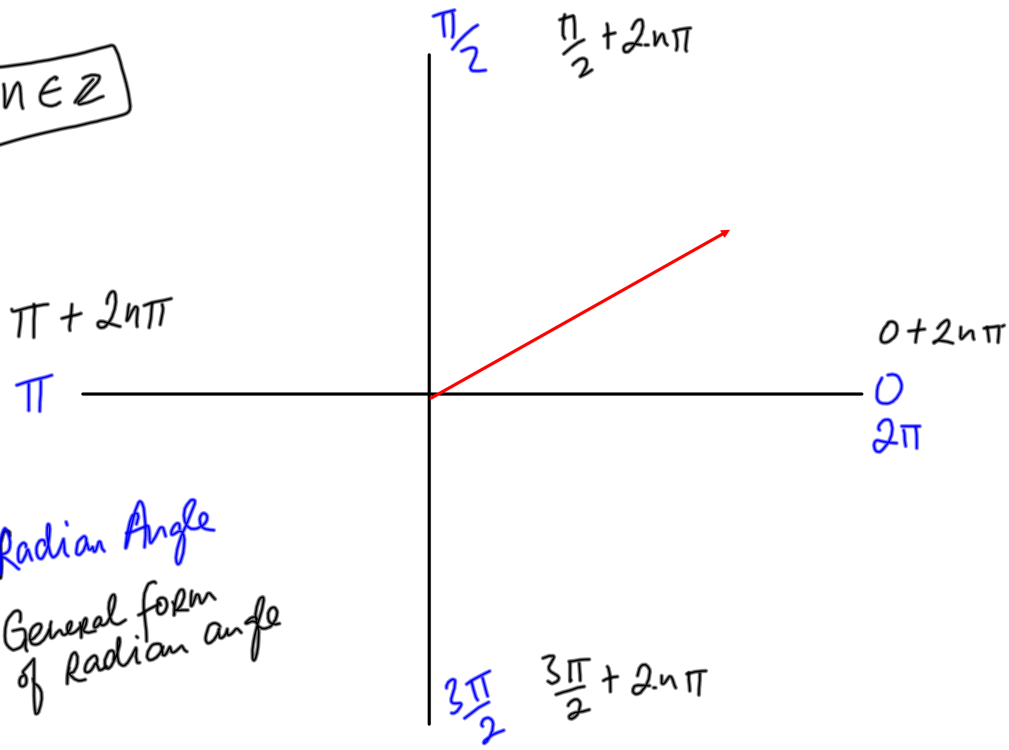
Expressing $\cos n\theta$ and $\sin n\theta$ in terms of $\cos \theta$ and $\sin \theta$

Example 2

Express (a) $\cos 2\theta$ in terms of $\cos \theta$ (b) $\sin 3\theta$ in terms of $\sin \theta$.

Trig

$\boxed{n \in \mathbb{Z}}$



Radian Angle

General form of Radian angle

Solve: $x^2 = 16$ $(-4)^2 = 16$

$x = 4$ or -4 Roots

Solve $x^3 = 27$ Root

$x = \sqrt[3]{27} = 3$

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not same as $x = 2$

$x^3 = ?$

$x^3 = 8$

How to find the nth root of a complex number

Given $z = a + bi$,
then $z = r[\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi)]$
where $n \in N$ is the **general polar form** of z.

Example 3

Solve the equation $z^3 = 8i$. $z = ?$ Roots?

$z = \sqrt[3]{8i} = (8i)^{\frac{1}{3}}$

$w = 0 + 8i$

$r = ?$ $\sqrt{a^2+b^2}$

$\theta = ?$

$r = \sqrt{0^2+8^2} = 8$

$8i$, Im, $\theta = \frac{\pi}{2}$, Re

Polar form $w = 8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

General Polar form $w = 8\left(\cos\left(\frac{\pi}{2}+2n\pi\right) + i\sin\left(\frac{\pi}{2}+2n\pi\right)\right)$

$z = w^{\frac{1}{3}}$ $z = w^{\frac{1}{3}} = \left[8\left(\cos\left(\frac{\pi}{2}+2n\pi\right) + i\sin\left(\frac{\pi}{2}+2n\pi\right)\right)\right.$

deMoivre $z = 8^{\frac{1}{3}}\left[\cos\frac{1}{3}\left(\frac{\pi}{2}+2n\pi\right) + i\sin\frac{1}{3}\left(\frac{\pi}{2}+2n\pi\right)\right]$

use $n=0$ $z_0 = 2\left[\cos\frac{1}{3}\left(\frac{\pi}{2}+2(0)\pi\right) + i\sin\frac{1}{3}\left(\frac{\pi}{2}+2(0)\pi\right)\right.$

$= 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \sqrt{3} + 1i$ *

$n=1$ $z_1 = 2\left[\cos\frac{1}{3}\left(\frac{\pi}{2}+2\pi\right) + i\sin\frac{1}{3}\left(\frac{\pi}{2}+2\pi\right)\right.$

* conjugates $= 2\left[\mathrm{cis}\,\frac{5\pi}{6}\right] = -\sqrt{3} + 1i$ *

$n=2$ $z_2 = 2\left[\mathrm{cis}\,\frac{1}{3}\left(\frac{\pi}{2}+4\pi\right)\right]$

$= 2\left[\mathrm{cis}\,\frac{3\pi}{2}\right] = 0 - 2i$

Check ANSWER IN EXAMPLE 3

Is $(-2i)^3 = 8i$?

$$(-2i)^3 = (-2i)^2(-2i)$$
$$= (\cancel{+}4\cancel{i^2})(-2i)$$
$$= 8i \quad \checkmark$$

If you expand $(\sqrt{3}+1i)^3$ or $(-\sqrt{3}+1i)^3$ you also get $8i$